Physics 218A

P. Diamond

Rayleigh – Taylor Instability ↔ A Case Study

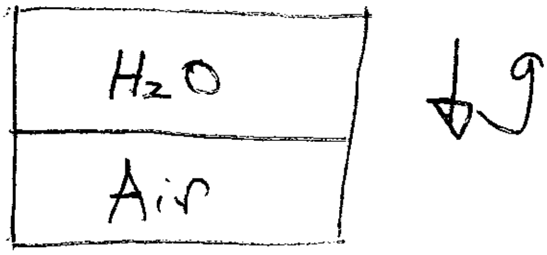
o.) Motivation and ICF Overview

→ RT is simple example/paradigm of non-trivial nonlinear collective dynamics

→ intellectual content typical of current problems in plasma physics → {nonlinear evolution of instabilities, turbulence, transport, etc.

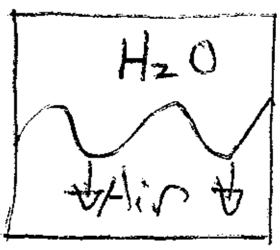
Overview of RT Physics:

I) Consider:



- free energy available (i.e. gravitational potential energy) (free energy ↔ instability ⁈) (successful storage ↔ confinement)
- system in equilibrium (i.e. inverted glass H_2O + cardboard) but small interface perturbations grow.

i.e.



water-glass demo.

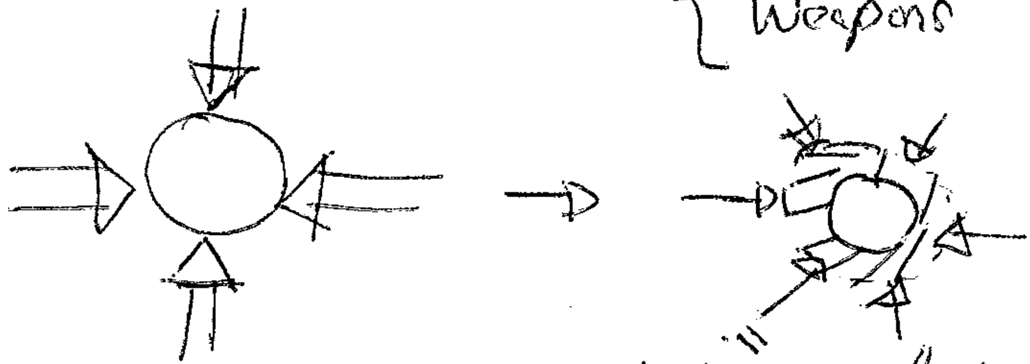
II) – typical evolutionary history:

[redacted] → instability occurs when light fluid accelerated <u>into</u> heavy fluid

⟺ in light fluid frame, equivalent to inverted water glass

I1 : Importance R-T in ICF !! $\underline{O}_{ps} \underline{1}$

e.g. spherical implosions { ICF, Weapons etc.



"hot light" fluid accelerated into "heavy" core

ablation-driven rocket

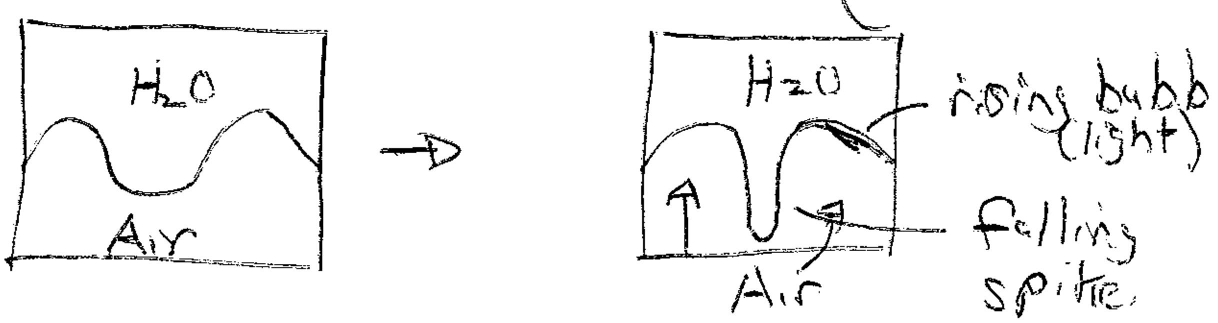
① → $\varepsilon < \lambda$ → linear growth phase

i.e. $\hat{\varepsilon}_{\underline{k}} = \hat{\varepsilon}_{\underline{k}}(0)\, e^{\gamma_{\underline{k}} t}$

↳ calculated from linear perturbation analysis

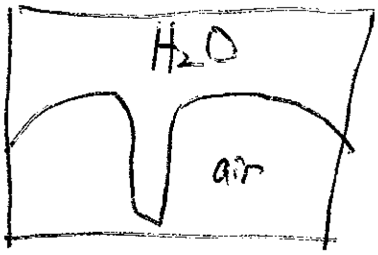
② → $\varepsilon \gtrsim \lambda$ → Spike and Bubble { Formation, Competition

i.e.

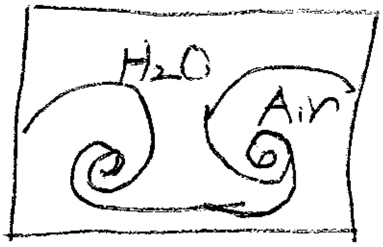


③ $\varepsilon \gtrsim \lambda \rightarrow$ Secondary Instability / Bubble competition

- Spike undergoes Kelvin-Helmholtz (shearing) instability
- Spike "rolls up" and is "blunted"

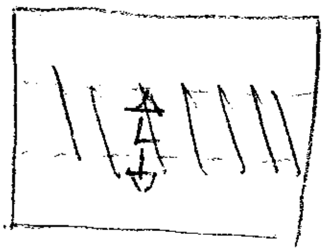


$\rightarrow$



④ $\varepsilon \gg \lambda \rightarrow$ Turbulent Mix

- Spike undergoes KH $\rightarrow$ turbulence generated
- Spike + bubble ensemble $\Rightarrow$ mixing layer, growing in time



phenomenological

$$L \sim \overset{\downarrow}{(.05)} \underbrace{\frac{(\rho_W - \rho_A)}{(\rho_W + \rho_A)} g t^2}_{\text{intuition from elementary mech}}$$

Note:

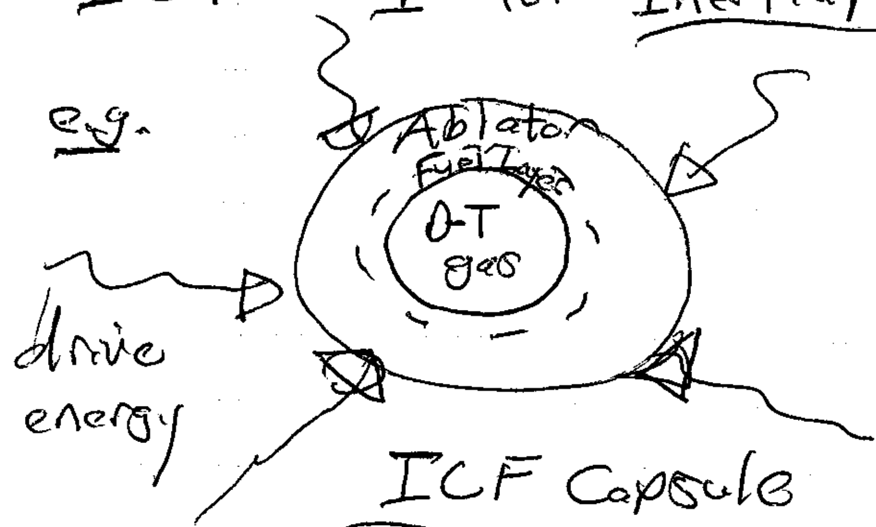
(i) Representation
- ① $\rightarrow$ Fourier Modes
- ②, ③ $\rightarrow$ Structures (Spike, Bubble)
- ④ $\rightarrow$ Turbulence

→ R.T. in I.C.F.

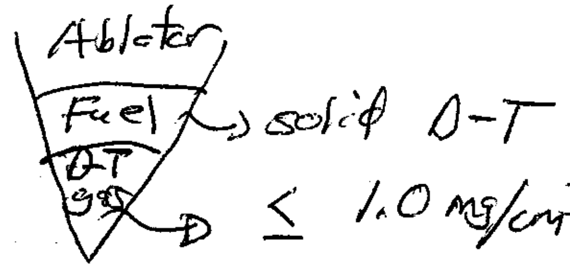
a.) Some Basics of ICF

ICF: I for Inertial

e.g.

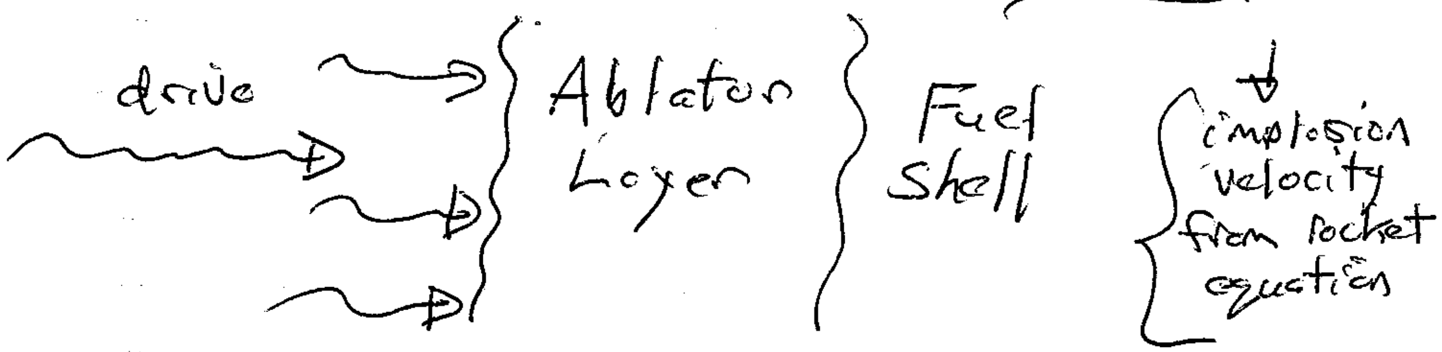


ICF Capsule

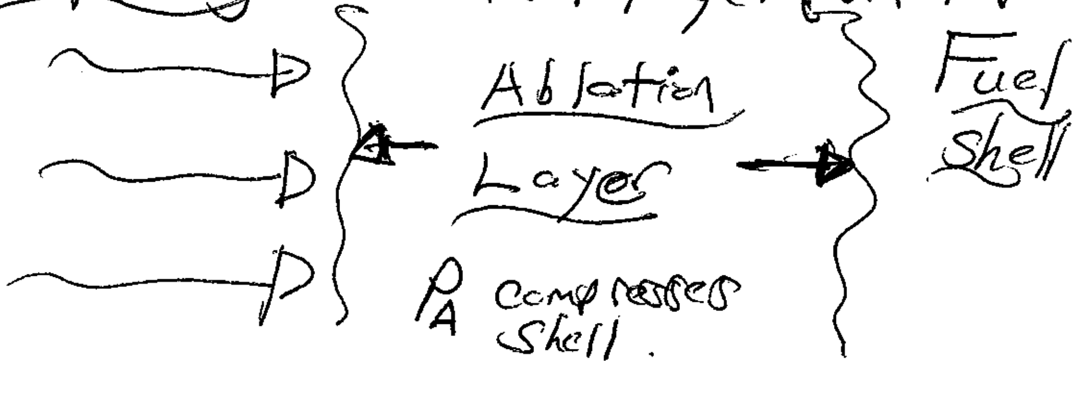


"drive" = laser or x-rays

How it works: Ablation-Driven Rocket



ablator layer heats and expands thus compressing inner fuel layer (via PV work)



note: → "implosion" is just conservation of momentum between expanding ablator layer and inner shell

→ $\underbrace{W_{OF} \text{ (work } \underline{\text{on}} \text{ fuel)}}$ $\sim P_A V_s$

P_A → ablation pressure

V_s → Vshell

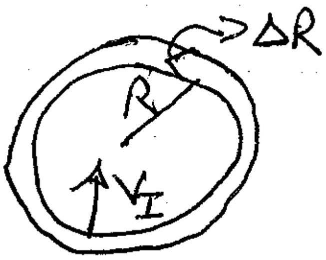
∴ for fixed P_A (determined by driver and materials), larger, thin shells can be accelerated better than small thick ones.

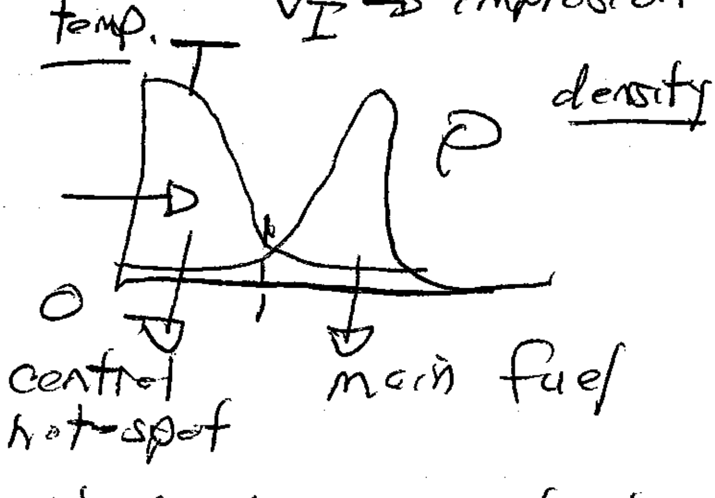
→ expected (hoped for...) final state eg:

R → shell radius

ΔR → shell thickness

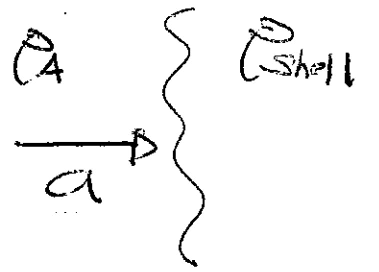
V_I → implosion velocity





idea is that burn initiates in central hot-spot, then propagates to main fuel shell.

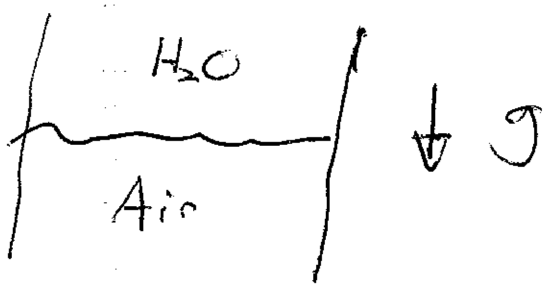
Now: Consider situation:



$\rho_{shell} > \rho_A$

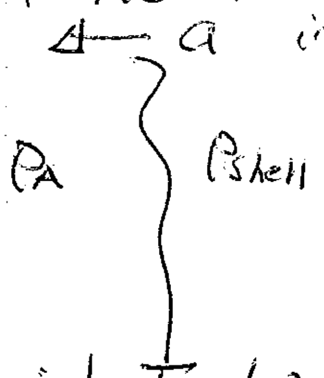
i.e. Light fluid "pushing on" (i.e. accelerating into) heavy fluid

Compare to inverted glass of H_2O:



$\rho_{H_2O} > \rho_{air}$

i.e. in frame of ablator interface, above:



$\Rightarrow$ Rayleigh Taylor Instability!

$\Rightarrow$ pgs. 1-2

Important features of Implosion:

→ IFAR – in flight aspect ratio
(→ stability)

$$\mathrm{IFAR} = \frac{R}{\Delta R}(t)$$

$\Delta R < \Delta R(t=0)$
due compr.

→ seek large IFAR

→ but R-T, constrains upper limit on IFAR → broadens ΔR via mixing
i.e. $25 < \mathrm{IFAR} < 35$

⇒ sets minimum P_A (~ 100 Mbar) and irradiance absorbed ($\sim 10^{15}$ W/cm²) for MJ drivers in order to achieve $V_I \sim 3\text{-}4 \times 10^7$ cm/sec.

⇔ R.T, is (partly) why NIF costs >1 BB i.e. drives cost of laser.

→ C_r – convergence ratio
(→ symmetry)

$$C_r = R_{A,i} / r_{\text{hot spot}, f}$$

i → initial
f → final

i.e. deviation from sphericity can destroy hot-spot (burn-thru), etc.

so $$\underbrace{\delta R}_{\text{deviation from sphericity}} = \tfrac{1}{2}\underbrace{\delta g}_{\text{deviation from avg. acceleration}} t^2 = \frac{\delta g}{g}(R_A - r) = \frac{\delta g}{g} r(C_r - 1)$$

Tolerable asymmetry $\Leftrightarrow$ excess of K.E. above ignition threshold. If demand, say

$$\delta R < \frac{r}{4} \Rightarrow \frac{\delta g}{g} \sim \frac{\delta V_I}{V_I} < \frac{1}{4(C_r - 1)}$$

since $C_r < 40$, need $\frac{\delta V}{V} \lesssim 1\%$!!

→ { Point is that R.T. $\Leftrightarrow$ ripples $\Leftrightarrow$ asymmetry, can destroy implosion via inducing asymmetry, unless $\underbrace{KE \gg \text{ignition threshold}}_{\text{Laser drive}}$ }

once again, R.T. $\Leftrightarrow$ \$

(ii) Evolution : ① → exponential

②, ③ → transition to algebraic

④ → algebraic

Skip, in favor II. II here

III) Application – ICF

Controlled Fusion ⟺ $n\tau T > (n\tau T)_{Lawson}$

Confinement → magnetic (tokamaks, etc.)

→ inertial (Laser acceleration, gravity (star))

→ ICF :

→ confine burning plasma via implosion driven by laser-produced ablation

→ implosion drives $n\tau T > (n\tau T)_{Lawson}$

Further :

→ optimal to implode <u>shell</u> :

acceleration (laser pulse) → outer surface ablated → RT unstable

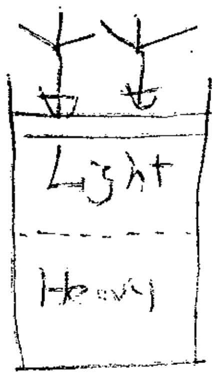
deceleration (post pulse) → inner gas accelerated into inner shell → RT unstable

→ implosion instability <u>intrinsic</u> to ICF

∴ Need understand, minimize

→ Basic Insight
- Computer Simulations
- Laboratory Experiments

Experimental Set-Up (Youngs Rocket Rig, D. Youngs, AWE)



→ Rocket Engine

Easy:

- diagnosis
- flow visulizations

References:

Landau, Lifshitz; Fluid Mechanics (Linear Theory)

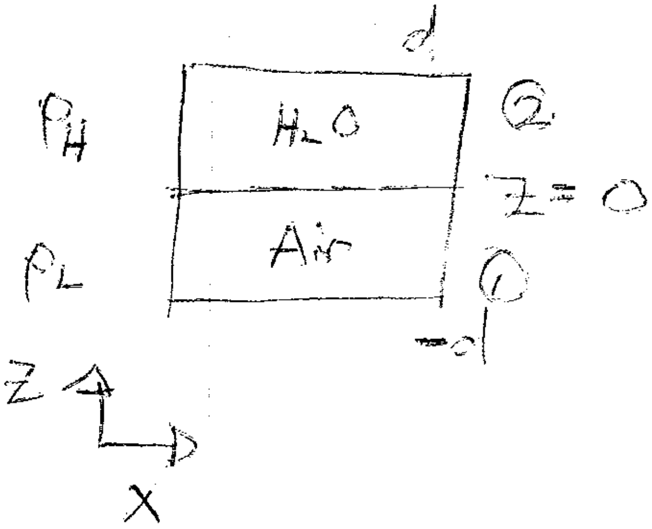
D. H. Sharp, Physica <u>12D</u> (1984) Pg. 3 (Overview)

S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability" Oxford U. Press (Linear Theory)

H. J. Kull, Physics Reports <u>206</u> #5 1991 (Review)

b.) Linear Theory

I) Hydrodynamic RT / Plane Slab



air

H_2O

→ surface wave.

Now, consider:

- incompressible fluid (i.e. $\gamma \ll kC_s$)

$$\nabla \cdot \underline{V} = 0$$

- irrotational flow $\nabla \times \underline{V} = \underline{\omega} = 0$
  (piecewise uniform density)

II → Newton's tube

$$\nabla \times \underline{V} = 0 \Rightarrow \underline{V} = \nabla \phi$$

stream function

$$\nabla \cdot \underline{V} = 0$$

$\Rightarrow \nabla^2 \psi = 0 \leftrightarrow$ R.T. instability is potential flow problem

Now, $\phi = \sum_{\underline{k}} \phi_{\underline{k}}(z)\, e^{ikx}$ (∞-ly wide or periodic box)

$\frac{\partial^2 \phi_{\underline{k}}(z)}{\partial z^2} - k^2 \phi_{\underline{k}} = 0$ $\Rightarrow$ origin of $\left\{ \begin{array}{l} p \\ \phi \checkmark \end{array} \right.$ continuity

For $kd \gg 1$, neglect finite depth, so

$$\phi_{\underline{k}} = \begin{cases} \phi_{\underline{k}}^{(1)}\, e^{kz} & z<0 \quad (1) \\ \phi_{\underline{k}}^{(2)}\, e^{-kz} & z>0 \quad (2) \end{cases}$$

(satisfy $V_n| = 0$ bndry)

At $z=0$:

$p^{(1)} = p^{(2)}$ $\rightarrow$ pressure continuity (else interface motion on accustic time scale)

$\left.\frac{\partial \phi^{(1)}}{\partial z}\right|_0 = \left.\frac{\partial \phi^{(2)}}{\partial z}\right|_0$ $\rightarrow$ normal velocity continuity

For dynamics:

$\rightarrow$ described entirely by interface motion

i.e.

$\rightarrow$ fields: $\eta(x, z, t)$ $\rightarrow$ <u>instantaneous</u> <u>interface position</u>

$\phi(x, z_i, t)$ $\rightarrow$ stream fcty

$$\downarrow$$

$z = 0 + \eta$

(η <u>dropped for linearized theory</u>) $\rightarrow$ why NLT hard.

For stream function: (Bernoulli's law)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \rho g \qquad (\underline{g} = -g\hat{z})$$

$$\underline{v} = \nabla\phi$$

$$\rho\left(\frac{\partial}{\partial t}\nabla\phi + \underline{\nabla\phi}\cdot\nabla\,\underline{\nabla\phi}\right) = -\nabla p - \rho g$$

$$\rho\left(\frac{\partial}{\partial t}\nabla\phi + \underline{\nabla}\left(\frac{\underline{\nabla\phi}\cdot\underline{\nabla\phi}}{2}\right)\right) = -\underline{\nabla} p - \rho g$$

$$\Rightarrow \boxed{\frac{\partial\phi}{\partial t} + \frac{\nabla\phi\cdot\nabla\phi}{2} = -\frac{p}{\rho} - g\eta} \qquad (\nabla = \nabla_h)$$

i.e. $\frac{\partial\phi}{\partial t} = 0$ $\Rightarrow$ $p + \frac{\rho V^2}{2} = \text{const.}$

$g = 0$

For interface:

$$\boxed{\frac{\partial\eta}{\partial t} + \underline{\nabla}\phi\cdot\nabla\eta = \frac{d\eta}{dt} = \frac{\partial\phi}{\partial z}} \longrightarrow \text{definition}$$

Then, linearizing for R.T. mode:

$$\frac{\partial\tilde{\phi}}{\partial t} = -\frac{\tilde{p}}{\rho} - g\tilde{\eta}$$

$$\frac{\partial\tilde{\eta}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial z}$$

Thus:

$$\rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t} + g\rho_2 \tilde{\eta} = -\tilde{p}^{(2)} \qquad (e^{-kz})$$

$$\rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} + g\rho_1 \tilde{\eta} = -\tilde{p}^{(1)} \qquad (e^{kz})$$

At interface: $\tilde{p}^{(1)}\Big|_0 = \tilde{p}^{(2)}\Big|_0$

$$\rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t} + g\rho_2 \tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} + g\rho_1 \tilde{\eta}$$

$$\tilde{V}_z^{(1)}\Big|_0 = \tilde{V}_z^{(2)}\Big|_0$$

$$\Rightarrow +k\tilde{\phi}^{(1)} = -k\tilde{\phi}^{(2)}$$

$\Rightarrow$

$$g(\rho_2-\rho_1)\tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} - \rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t}$$

$$= (\rho_1+\rho_2) \frac{\partial \tilde{\phi}^{(1)}}{\partial t}$$

$\therefore$

$$\frac{\partial \tilde{\phi}^{(1)}}{\partial t} = g\frac{(\rho_2-\rho_1)}{(\rho_1+\rho_2)}\tilde{\eta}$$

$$\frac{\partial \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}^{(1)}}{\partial z}$$

$$\frac{\partial^2 \tilde{\phi}^{(1)}}{\partial t^2} = g \frac{(\rho_2-\rho_1)}{(\rho_2+\rho_1)} \frac{\partial \tilde{\phi}^{(1)}}{\partial z}$$

$\Rightarrow$

$$\omega_1^2 = -gAk$$

$$\boxed{\gamma = \sqrt{gA}\,\sqrt{k}}$$

$\left\{ \begin{array}{l} A \equiv \dfrac{\rho_2-\rho_1}{\rho_2+\rho_1} \\ \text{Atwood \# - available free energy} \end{array} \right.$

Comments:

i.) equivalent: $\left\{ \begin{array}{l} \text{fluid with} \\ \text{vacuum} \end{array} \right.$ $\rho \rightarrow A$

ii.) H_2O, air: $\lambda = 1\,cm$ $\quad \gamma^{-1} \sim .1$ sec! (fast)

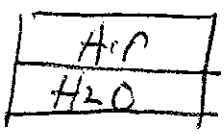
iii.) $\gamma = \sqrt{gAk}$

$\therefore$ in absence dissp., surface tension etc., shorter wavelengths grow faster

iv.) $A<0 \Rightarrow$ stable stratification

$\rightarrow$ surface buoyancy wave

H_2O, Air $\Rightarrow \omega = \sqrt{kg}$ $\rightarrow$ surface gravity wave



-.) Other Effects:

i.) Surface Tension (Fluid) → I III   (HW)

- curvature of interface exerts force

i.e. $p \to p - \rho \gamma_T \nabla_h^2 \eta$ $\left(\gamma_T = \dfrac{T_s}{\rho}\right)$

(For H_2O - air, only H_2O feels surface tension; for fluid ①, fluid ②, T_s for each interface)

$$\Rightarrow \quad \gamma = \left(kgA - \gamma_T k^3\right)^{1/2}$$

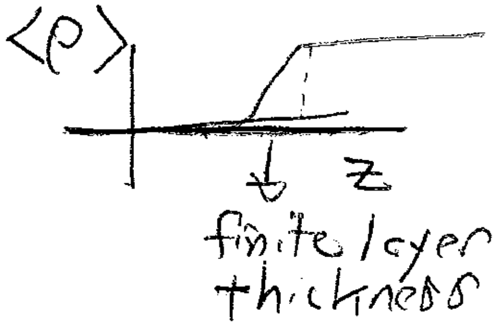
$$k_{max}\Big|_{\text{unstable}} = \left(gA/\gamma_T\right)^{1/2}$$

→ range of modes limited

e.g. inverted glass with cardboard → $\gamma_T \to \infty$

ii.) Finite Interface Thickness - $\nabla \rho$

i.e.



Consider opposite limit:

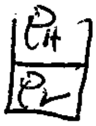
$k L_\rho \gg 1$ $L_\rho^{-1} = 1/\rho_0 \; d\bar{\rho}/dz$ (rippled interface → cell)

- fluid motion not irrotational, as $\nabla \rho \nparallel \nabla p$

hydrostatic eqbrm $\frac{dp}{dz} = -\rho g$

Review

→ Last time:



$$\nabla^2\phi = 0 \Rightarrow \begin{cases} \phi_H = \tilde{\phi}_H e^{-kz} \\ \phi_L = \tilde{\phi}_L e^{kz} \end{cases}$$

$$\begin{cases} \dfrac{\partial\phi}{\partial t} + \dfrac{(\nabla\phi)^2}{2} = -\dfrac{p}{\rho} - g\eta & \rightarrow \text{Bernoulli} \\[2ex] \dfrac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \dfrac{\partial\phi}{\partial z} & \rightarrow \text{defn.} \end{cases}$$

$$\tilde{V}_{Hz} = \tilde{V}_{Lz} \qquad \tilde{p}_H = \tilde{p}_L$$

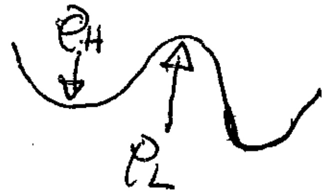
LT.

$$\Rightarrow \quad \gamma = \sqrt{gAk_\perp} \qquad \left\{ A = \left[\frac{\rho_H - \rho_L}{\rho_H + \rho_L}\right] \right.$$

→ key Assumptions:

easy $\begin{cases} \text{- incompressible} & \rightarrow \gamma \lll k_\perp c_s \\ \text{- inviscid} & \rightarrow \gamma \gg \nu k_\perp^2 \end{cases}$

- irrotational → - thin interface
$\underline{V} = \nabla\phi$ ("piecewise uniformity")
- no breaking → {amplitude restricted
→ potential flow. - no K.H.

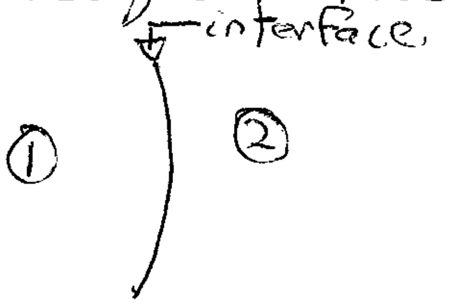
i.e. 

→ interface ripples

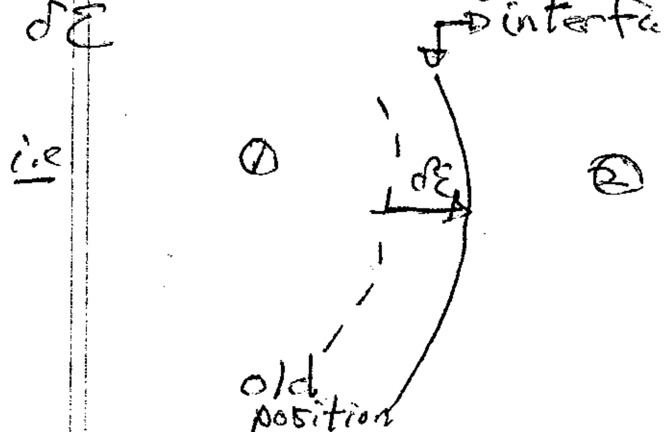
but "heavy" falls
light "rises"

Insert III: Surface Tension

→ Consider two liquids separated by a thin (i.e. few molecules) interface



Now, consider displacing the interface toward ② by $\delta\xi$



$(\xi \leftrightarrow \lambda)$

∴ can determine change in free energy (i.e. thermodynamic sense) via:

$$dF = \underbrace{dF_{①} + dF_{②}}_{\text{bulk phases}} + dF_{\text{interface}}$$

↳ treat as separate constituent

Recall: $dF = -SdT - pdV$ (i.e. $F = E - ST$)

∴ $dF_{①,②} = (-SdT - pdV)_{①,②}$

(i.e. the usual)

then, for interface, natural to define:

$$dF_I = -S_I dT + \sigma dA,$$

S_I: entropy of interface

σdA: change in free energy due increase in surface area of interface (treat as separate phase)

$\sigma \equiv$ Surface Tension
($\sim$ Pressure $\times$ Length $\sim$ Force/Length)
(instead F/Area $\times$ L)

Hereafter, consider isothermal displacement.

$$\rightarrow \quad dF = -p_1 dV - p_2(-dV) + \sigma dA$$

$$= (p_2 - p_1) dV + \sigma dA$$

interface expands 'into' 2nd material

Furthur: $dV = dA\, d\xi$ (for surface)

$d\xi$: displacement, $\xi(x,y)$

For dA: $dA = \int dx\,dy \left(1 + \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right)^{1/2} - \int dx\,dy$

.. for small displacement:

$$dA \cong \int dx\,dy \left(\cancel{1} + \frac{1}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \xi}{\partial y}\right)^2\right)$$

$\therefore \quad dA = \int dx\,dy\,\underbrace{(-\nabla^2 \xi)}_{\text{curvature of surface displacement}}\, d\xi$ (i.e. anticipates integration by parts)

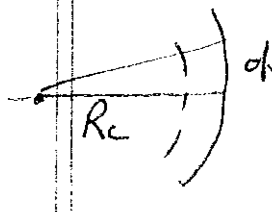
so $\quad dF = \int \left[(P_2 - P_1)\, dA_0 - \sigma \nabla^2 \xi \, dA_0 \right] \delta \xi$

$\Rightarrow$ condition for equilibrium:

$$P_2 - P_1 = \sigma \nabla^2 \xi(x,y)$$

More generally: $\quad dF = (P_2 - P_1)\, dA_0\, d\xi + \sigma\, dA$

Now consider arbitrary (i.e. not weakly curved interface)



$$ds' = (R_c + d\xi)\, d\theta$$
$$= dl_0 \left(1 + \frac{d\xi}{R_1} \right)$$

In general, surface parametrized by $\underline{2}$ radii of curvature R_1, R_2

so $\quad dA = \int dl_1\, dl_2 \left(1 + \frac{d\xi}{R_1} \right) \left(1 + \frac{d\xi}{R_2} \right) - \int dl_1\, dl_2$

$$dA = \int dl_1 \, dl_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) d\Sigma$$

Thus, have <u>most</u> <u>general</u> expression:

$$dF = \int \left[(P_2 - P_1)\, dA_0 - \left(\frac{1}{R_1} + \frac{1}{R_2} \right) dA_0 \right] d\Sigma$$

thus, for equilibrium with interface:

$$\boxed{\sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = -(P_2 - P_1)}$$

Laplace's Law

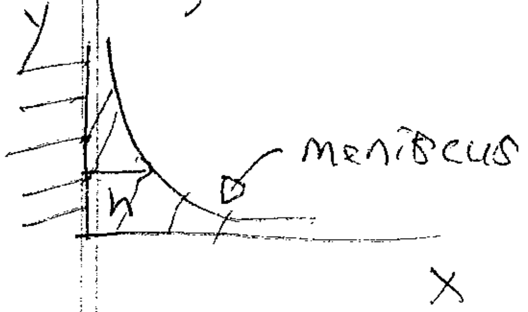
<u>i.e.</u> → given 2-phase equilibrium (separated)
can use to estimate droplet size for immiscible liquids

<u>i.e.</u> if $P_2 < P_1$

∴ droplets of size $R \sim \sigma / (P_1 - P_2)$ may be expected.

↓ skip to I5

→ consider liquid adjacent to fixed vertical wall, then:



$h(y) \equiv$ defines thickness of meniscus

Then, can write:

known

$$p_{liq} = p_0 - \rho g\, y(x) \qquad (g<0)$$

to calculate $h(y(x))$, use Laplace's Law:

i.e. $$p_0 - \rho g y = \frac{\sigma}{R_c}$$

but $$\frac{1}{R_c} = \frac{-\partial^2 h/\partial x^2}{\left(1+(\partial h/\partial x)^2\right)^{3/2}}$$

(i.e. don't make small curvature approx)

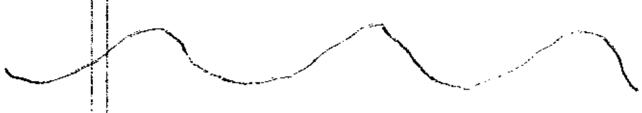
then taking $p_0 = 0$ (ref):

$$+\rho g\, y(x) = + \frac{\partial^2 h(y)/\partial x^2}{\left(1+(\partial h/\partial x)^2\right)^{3/2}}$$

and can get dh/dy, etc.

* → Capillary Waves.

Recall discussed ocean waves (stable R.T.)



Should be apparant now that:

→ for high k, curvature of crests, etc. becomes sharp

→ before, tacitly took $\rho g \eta \gg \frac{\sigma}{R_c}$

now if $R_c \sim \eta$ s/t $\eta^2 \sim \sigma / \rho g$

<u>must</u> retain surface tension in surface wave dynamics ⇒ capillary waves

To include:

$$p = p_0 - \sigma \nabla^2 \eta$$

Then recall: $$\frac{\partial \tilde{\phi}}{\partial t} = -\frac{\tilde{p}}{\rho} - g\tilde{\eta}$$

$$\frac{\partial \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial z}$$

$\therefore$ $$\frac{\partial \tilde{\phi}}{\partial t} = \frac{\sigma}{\rho} \nabla^2 \tilde{\eta} - g\tilde{\eta}$$

$$\frac{\partial \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial z}$$

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} = \frac{\sigma}{\rho} \nabla^2 \frac{\partial \tilde{\phi}}{\partial z} - g \frac{\partial \tilde{\phi}}{\partial z}$$

$$\Rightarrow \boxed{\omega^2 = kg + \frac{\sigma k^3}{\rho}}$$ → dispersion relation for capillary wave

note: – capillarity estimate $l \sim \sigma/\rho g$

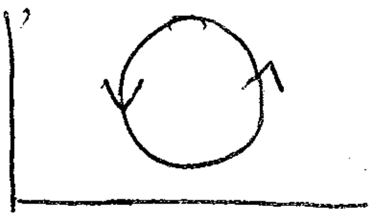
d.r. $\Rightarrow k_{cap}^2 \sim \rho g/\sigma$ ✓

– in ocean, capillarity significant at $\lesssim 5c$

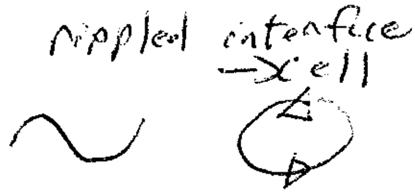
– if R.T. unstable, capillarity will cut-off high k_i instability

i.e. $\omega^2 = \frac{-kg(\rho_2-\rho_1)}{(\rho_2+\rho_1)} + \frac{\sigma \tilde{k}^3}{\rho_2+\rho_1}$

- motion is that of convective cells, vortices

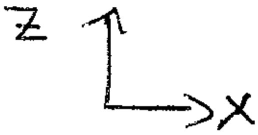


$\downarrow g \quad \uparrow \rho$



To calculate:

- For 2D cell



$$\frac{\partial \tilde{V}_x}{\partial t} = -\partial_x\left(\frac{p}{\rho_0}\right) \qquad\qquad \frac{\partial \tilde{\rho}}{\partial t} = -\tilde{V}_z \frac{d\rho_0}{dz}$$

$$\frac{\partial \tilde{V}_z}{\partial t} = -\partial_z\left(\frac{p}{\rho_0}\right) - g\frac{\tilde{\rho}}{\rho_0}$$

Suggests write: $\underline{V} = \underline{\nabla}\phi \times \hat{y}$

$$\Rightarrow \quad \tilde{V}_x = -\partial_z \tilde{\phi}$$
$$\tilde{V}_z = \partial_x \tilde{\phi}$$

$$-\frac{\partial}{\partial t}\partial_z\tilde{\phi} = -\partial_x\left(\frac{p}{\rho_0}\right) \qquad (1)$$

$$+\frac{\partial}{\partial t}(\partial_x\tilde{\phi}) = -\partial_z(p/\rho_0) - g\frac{\tilde{\rho}}{\rho_0} \qquad (2)$$

$\partial_z(1) - \partial_x(2) \Rightarrow$

$$-\frac{\partial}{\partial t}\nabla^2\tilde{\phi} = \frac{\partial}{\partial x}\left(g\frac{\tilde{\rho}}{\rho_0}\right)$$

$-\nabla^2\phi = \omega_y$
$\downarrow$
$\hat{y}$ component
vorticity

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \nabla^2 \tilde{\phi} &= -\frac{\partial}{\partial x}\left(g \tilde{\rho}/\rho_0\right) \\ \frac{\partial}{\partial t} \tilde{\rho} &= -\partial_x \tilde{\phi} \frac{d\rho_0}{dz} \end{aligned}}$$

$$\frac{\partial^2}{\partial t^2} \nabla^2 \tilde{\phi} = \left(\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right) \frac{\partial^2 \tilde{\phi}}{\partial x^2}$$

$$\Rightarrow \quad +\omega^2 k^2 = \left(\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right)\left(-k_x^2\right)$$

$$\boxed{\omega^2 = -\frac{k_x^2}{k^2}\left(\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right)}$$

$\hookrightarrow > 0$, as $d\rho_0/dz > 0$

$$\therefore \quad \gamma = \sqrt{\frac{k_x^2}{k^2}}\left(\frac{g}{L_\rho}\right)^{1/2} \rightarrow \text{R.T. Convective cell growth-rate}$$

Then:

- structure similar to Rayleigh - Benard convection

i.e. $\frac{\partial}{\partial t}$ vorticity = torque
- buoyancy (RB)
- gravitational force (RT)

→ $k_x \to \infty \Rightarrow \gamma \to \frac{g}{L_\rho}$

Thus, to incorporate finite interface thickness in RT growth formula

$$\gamma \sim \sqrt{gAk} \qquad kL_\rho < 1$$
$$\sim \sqrt{g/L_\rho} \qquad kL_\rho > 1$$

$$\Rightarrow \gamma = \left(gAk/1 + \underset{\substack{\downarrow \\ \text{scale factor, interface.}}}{kL}\right)^{1/2}$$

∴ $kL > 1 \Rightarrow$ growth rate saturates!

→ For stable stratification $\quad d\rho_0/dz < 0$

$$\omega^2 = \frac{k_x^2}{k^2} \frac{g}{\rho_0} \left|\frac{d\rho_0}{dz}\right| \equiv \frac{k_x^2}{k^2} N^2$$ ↳ BV freq

→ dispersion relation for oceanic internal wave

→ finite density gradient analogue of (interface) surface wave

– interesting to note effects of

viscosity
particle diffusivity

viscosity $\quad \frac{\partial}{\partial t}\nabla^2\phi \rightarrow \left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\phi$

diffusivity $\quad \frac{\partial}{\partial t}\rho \rightarrow \left(\frac{\partial}{\partial t} - D\nabla^2\right)\rho$

$\Rightarrow$

$$(\omega + i\nu k^2)(\omega + iDk^2) = -\frac{k_x^2}{k^2}\frac{g}{\rho_0}\frac{d\rho_0}{dz}$$

i.e.

① $\begin{cases} \nu k^2 \gg \omega \\ D \to 0 \end{cases}$ (viscous fluid)

$$(i\nu k^2)(i\gamma) = -\frac{k_x^2}{k^2}\frac{g}{\rho_0}\frac{d\rho_0}{dz}$$

$$\gamma = \frac{k_x^2}{k^2}\left(\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right) \Big/ \nu k^2$$

$\rightarrow \gamma \sim 1/\nu k^2$

$\rightarrow$ strong viscosity reduces growth rate but instability persists

(i.e. mollasses + air!)

i.e.

② $D = \nu$

$$\gamma = \left(\frac{k_x^2}{k^2}\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right)^{1/2} - \nu k^2$$

i.e. viscosity and diffusivity can stabilize R T instability

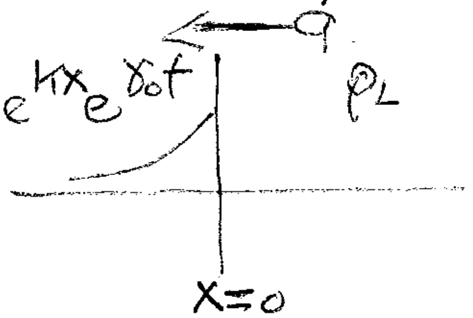
→ defines critical $\nabla\rho$

i.) Ablation (Ablation critical element of environment implosion ⇒ ablation driven rocket)

→ physical concept is that due to heating, material streams away from interface, ∴ can't participate in R.T. instability

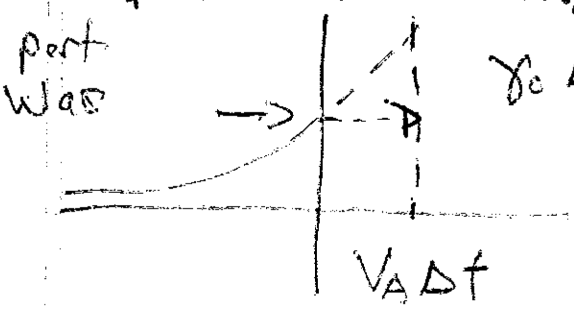
→ heuristic interpretation:

i.e. $\phi \sim e^{kx} e^{\gamma_0 t}$



e^{+kx} $(x<0)$ is wave-function shape.

with ablation, hot matter "blown off" ⇒ interface displaced inward



$\gamma_0 \Delta t - k V_A \Delta t$

i.e. $e^{k(x - \underbrace{V_A \Delta t}_{\text{re-absorb}})}$

burn-off ⇒ interface moves inward

i.e. $\delta\phi \sim e^{k(x - V_A \Delta t)} e^{\gamma_0 \Delta t}$

$\sim e^{kx} e^{(\gamma_0 - k V_A)\Delta t}$

$V_{Abl} \equiv \dfrac{\dot{M}}{\rho_A A}$

∴ ablative blow-off yields stabilizing effect

$\gamma = \gamma_0 - \underset{\text{fct \#}}{k} V_A$; $\gamma_0 = \sqrt{kg}$

$\Rightarrow$ no simple, rigorous analytical theory exists!

Aside: For ICF, can combine finite interface thickness and ablative stabilization to control RT growth (A=1)

i.e. simple RT $\quad \gamma = \sqrt{kg}$

finite interface $\Rightarrow \gamma = \left(kg / 1+kL_p \right)^{1/2}$

ablation $\Rightarrow \gamma = \left(kg / 1+kL_p \right)^{1/2} - kV_A$

By
- target design (structure) $- L_p$
- materials, etc (doping) $- V_A$

} can minimize implosion pert. growth

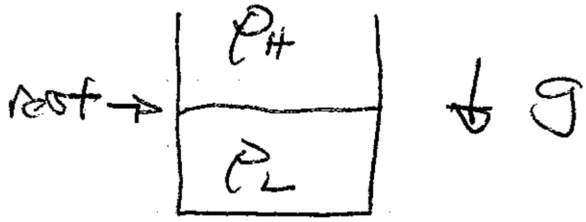
(V.) Spherical Geometry – Postpone till later

Crudely: $\begin{cases} \omega \sim g/u \\ u \sim \# \sqrt{g\lambda} \end{cases}$

N.B.: $\left\{ \begin{array}{l} \text{Can solve 3 bubble Layzer} \\ \text{model (numerically) to determine \#} \\ \text{in merger rule.} \end{array} \right.$

→ HW on Tuesday
→ no note borrowing on Tues, Th. A.M.

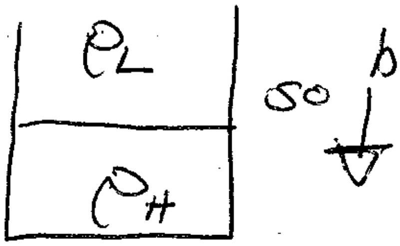
Note equivalence :



to :

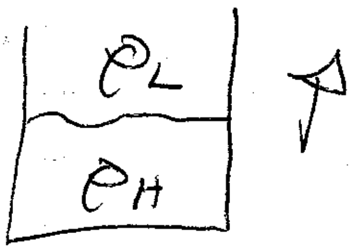
but $\rho_L >> \rho_H$

g ignored.



so a_{eff}.

in frame of interface/membrane, equivalent to



$g_{eff} = -a_{eff}$.
(same as pedagogical example)

In general :

"R-T" occurs when light fluid 'accelerated into/toward' heavy fluid.